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Corrections to "On certain homeomorphism groups"

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Recently, we noted that the inductive step in the proof of Lemma 4 [1] was incomplete in the case $G = G_0(\mathbb{R}^n)$, $n \geq 1$. Here we give a different proof of that lemma for $G_0(\mathbb{R}^n)$, $n > 1$. For $n = 1$, we are able to obtain only a slightly weaker form of the lemma. However, this weaker form is all that is needed in the proof of Theorem 5 [1] and so the results of our paper remain valid. We shall preserve the notations of our paper. Thus $G_0(\mathbb{R}^n)$ denotes the group of all homeomorphisms of $\mathbb{R}^n$ with compact supports.

Let $n \geq 2$. The group $G = G_0(\mathbb{R}^n)$ has the following property: given any sequence of pairwise distinct points $x_1, \ldots, x_k, y_1, \ldots, y_k$, k any positive integer, there exists a $g \in G$ such that $g(x_i) = y_i$, for $1 \leq i \leq k$. This property fails for $G_0(\mathbb{R})$. However, if $x_1 < y_1 \cdots < x_k < y_k$ in $\mathbb{R}$, then there exists a $g \in G_0(\mathbb{R})$ such that $g(x_i) = y_i$, $1 \leq i \leq k$. In fact, one could find a $g \in G_0(\mathbb{R})$ such that g is piecewise linear in $\mathbb{R}$ having support in any prescribed interval (x, y), where $x < x_1 < y_k < y$.

For $n \geq 1$, let $\mathscr{B}_n$ denote the collection of all open sets in $\mathbb{R}^n$ of the form $J_1 \times \cdots \times J_n$ where each J_i is a nonempty bounded interval in $\mathbb{R}$ with rational end points. For $U, V \in \mathscr{B}_1$ we write $U < V$ to mean $\bar{U} \cap \bar{V} = \emptyset$, and $x < y$ if $x \in U$ and $y \in V$.

Lemma 1. *Let $n \geq 1$, $k \geq 1$. Let $U_1, V_1, \ldots, U_k, V_k \in \mathscr{B}_n$ be such that $\bar{U}_i \cap \bar{U}_j = \bar{U}_i \cap \bar{V}_p = \bar{V}_i \cap \bar{V}_j = \emptyset$ for $1 \leq i, j, p \leq k$, $i \neq j \neq p$. When $n = 1$, assume also that $U_1 < V_1 < \cdots < U_k < V_k$. Then, given any sequence of nonzero integers $n_1, \ldots, n_k$, there exists a $g \in G_0(\mathbb{R}^n)$ such that $g^{n_i}(U_i) \subset V_i$ for $1 \leq i \leq k$.*

Proof. This is easy for $k = 1$. In fact, one can find an element g_i in $G_0(\mathbb{R}^n)$ with support contained in an open set W_i, $1 \leq i \leq k$ such that $g_i^{n_i}(U_i) \subset V_i$ and $\bar{W}_i \cap \bar{W}_j = \emptyset$, for $i \neq j$. Then it is straightforward to verify that $g = g_1, \ldots, g_k \in G_0(\mathbb{R}^n)$, and $g^{n_i}(U_i) \subset V_i$, $1 \leq i \leq k$. □

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For each $k \geq 1$, and each sequence of open sets $U_1, V_1, \ldots, U_k, V_k \in G_0(\mathbb{R}^n)$, satisfying the hypothesis of Lemma 1, and each sequence of nonzero integers $n_1, \ldots, n_k$, choose a $g \in G_0(\mathbb{R}^n)$ such that $g^{n_i}(U_i) \subset V_i$, $1 \leq i \leq k$, and fix it. Let E_n denote the subset of $G_0(\mathbb{R}^n)$ of such chosen g. Note that the set E_n is countable.

The following is a restatement of Lemma 4 of [1] for $n > 1$.

Lemma 2. *Let $n \geq 1$. Let $h_1, \ldots, h_k \in G_0(\mathbb{R}^n)\backslash\{1\}$, and let $n_1, \ldots, n_k$ be a sequence of non- zero integers. When $n = 1$, assume further that there exists $x_i \in \mathbb{R}$ such that $x_1 < y_1 < \cdots < x_k < y_k$, where $h_i(x_i) = y_i$, $1 \leq i \leq k$. Then there exists a $g \in G_0(\mathbb{R}^n)$ such that*

$$h_k g^{n_k} \cdots h_1 g^{n_1} \neq 1.$$

Proof. First, choose points $u_i \in \mathbb{R}^n$, $1 \leq i \leq k$ such that $h_i(u_i) = v_i \neq u_i$, with $u_1, v_1, \ldots, u_k, v_k$ being pairwise distinct. In case $n = 1$, choose $u_i = x_i, v_i = y_i$, where x_i, y_i are as in the statement of the lemma. Then one can find neighbourhoods $U_1, V_1, \ldots, U_k, V_k \in \mathscr{B}_n$, $v_i \in V_i$, $u_i \in U_i$ such that $\bar{U}_i \cap \bar{U}_j = \bar{V}_i \cap \bar{V}_j = \bar{U}_i \cap \bar{V}_p = \emptyset$ for $1 \leq i, j, p \leq k$, $i \neq j \neq p$. In case $n = 1$ we may assume that $U_1 < V_1 < \cdots < U_k < V_k$.

Now, choose a $V_0 \in \mathscr{B}_n$ whose closure is disjoint from $\bigcup_{1 \leq i \leq k} (\bar{U}_i \cup \bar{V}_i)$. When $n = 1$, we can arrange so that $V_0 < U_1$. Now applying Lemma 1 we see that there exists a g in E_n such that $g^{n_i}(V_{i-1}) \subset U_i$, $1 \leq i \leq k$. A routine verification now shows that $h_k g^{n_k} \cdots h_1 g^{n_1}$ maps V_0 into V_k. It follows that $h_k g^{n_k} \cdots h_1 g^{n_1} \neq 1$. □

When $n = 1$, the hypothesis of the above lemma does not hold for an arbitrary finite sequence of elements $G_0(\mathbb{R})$. Therefore, the above lemma does not imply Lemma 4 of [1]. However, we show below that one can find a copy H of $G_0(\mathbb{R})$ in $G_0(\mathbb{R})$ so that the hypothesis of the above lemma does hold for any finite sequence of elements of H.

Let $J_1 < K_1 < \cdots < J_m < K_m < \cdots$ be a sequence of intervals in $\mathbb{R}$, $J_m, K_m \in \mathscr{B}_1$ such that no two of them have a common end point, and $\lim J_m = p = \lim K_m$ for some $p \in \mathbb{R}$ (that is $\lim a_m = p = \lim b_m = p$ for any $a_m \in J_m, b_m \in K_m$). Choose order preserving homeomorphisms $\phi_m : J_m \to \mathbb{R}$ and *order reversing* homeomorphisms $\theta_m : K_m \to \mathbb{R}$ for each $m \geq 1$.

Let $m \geq 1$. Given any $h \in G_0(\mathbb{R})$, let $\psi_m(h) : \mathbb{R} \to \mathbb{R}$ be the map

$$\psi_m(h)(x) = \begin{cases} \phi_m^{-1} h \phi_m(x) & \text{if } x \in J_m, \\ \theta_m^{-1} h \theta_m(x) & \text{if } x \in K_m, \\ x & \text{if } x \notin (J_m \cup K_m). \end{cases}$$

It is straightforward to verify that each $\psi_m : G_0(\mathbb{R}) \to G_0(\mathbb{R})$ is a monomorphism of groups, and that for any $h \neq 1$, in $G_0(\mathbb{R})$, there exists an $x \in J_m \cup K_m$ such that $\psi_m(h)(x) > x$.

Finally let $\psi : G_0(\mathbb{R}) \to G_0(\mathbb{R})$ be defined as $\psi(h) = \prod_{m \geq 1} \psi_m(h)$. More precisely,

$$\psi(h)(x) = \begin{cases} \psi_m(h)(x) & \text{if } x \in J_m \cup K_m, \\ x & \text{if } x \notin \bigcup_{m \geq 1} (J_m \cup K_m). \end{cases}$$

Then the map ψ is a well-defined monomorphism of groups. Given any sequence $h_1, \dots, h_k \in G_0(\mathbb{R})\backslash\{1\}$, we choose $x_j \in J_j \cup K_j$, $1 \le j \le j \le k$ such that $y_j := \psi(h_j)(x_j) = \psi_j(h_j)(x_j) > x_j$, $1 \le j \le k$. Then we have $x_1 < y_1 < \cdots < x_k < y_k$. This proves the following assertion.

Lemma 3. *There exists a monomorphism of groups*

$$\psi : G_0(\mathbb{R}) \to G_0(\mathbb{R})$$

such that for any $h_1, \dots, h_k \in G_0(\mathbb{R})\backslash\{1\}$, the sequence of elements $\psi(h_1), \dots, \psi(h_k)$ satisfies the hypothesis of Lemma 1. In particular, given any sequence of nonzero integers $n_1, \dots, n_k$, there exists a $g \in E_1$ such that

$$\psi(h_k)g^{n_k} \cdots \psi(h_1)g^{n_1} \neq 1.$$

As observed earlier, Lemma 4 of [1] for the case $G_0(\mathbb{R}^n)$, $n \ge 2$, follows from Lemma 1, and in fact, replacing E by E_n, $n \ge 2$, we see that we can avoid the use of Schreier–Ulam metric on $G_0(\mathbb{R}^n)$. In case of $G = G_0(\mathbb{R})$, we need only replace the diagonal copy $\mathscr{G}$ of G in G^ω in the proof of Theorem 5 [1] by the image of G under the composite

$$G \xrightarrow{\psi} G \xrightarrow{\Delta} G^\omega.$$

Here again one can take E to be E_1. Then the rest of the proof of Theorem 5 [1] goes through for this copy of G, showing $G * \mathbb{Z} \simeq \Delta \circ \psi(G) * \mathbb{Z}$ embeds in G.

Addendum: We wish to add the following corollary to Theorem 5 [1].

Theorem 4. *Let G denote any one of the following groups: $G_0(\mathbb{R}^n)$, $n \ge 1$; the group of all homeomorphisms of X, where $X = \mathbb{Q}$, $\mathbb{N}$, the space of irrational numbers, the Cantor set; or the group of all order preserving homeomorphisms of the reals. Let $G_\lambda = G$ for $\lambda \in \mathbb{R}$. Then the free product $*_{\lambda \in \mathbb{R}} G_\lambda$ embeds in G.*

Proof. Let $\{t_\lambda\}_{\lambda \in \mathbb{R}}$, denote a set of free generators of a free group F of rank the continuum. By Theorem 5 [1], we know that $G * F$ embeds in G. Now each subgroup $t_\lambda^{-1} G t_\lambda$ of $G * F$ is isomorphic to G, and they generate their free product in $G * F$ as λ varies in $\mathbb{R}$. As $G * F$ embeds in G, this proves the theorem. □

References

[1] P. Sankaran and K. Varadarajan, On certain homeomorphism groups, J. Pure Appl. Math. 92 (1994) 191–197.